

# Theoretical Physics Institute

## University of Minnesota

TPI-MINN-01/10-T

UMN-TH-1942-01

February 2001

## Once again on electromagnetic properties of a domain wall interacting with charged fermions

**M.B. Voloshin**

Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455

and

Institute of Theoretical and Experimental Physics, Moscow, 117259

### Abstract

The response to a magnetic flux is considered of the vacuum state of charged Dirac fermions interacting with a domain wall made of a neutral spinless field in  $(3+1)$  dimensions with the fermion mass having a phase variation across the wall. It is pointed out that due to simple  $C$  parity arguments the spontaneous magnetization for this system is necessarily zero, thus invalidating some claims to the contrary in the literature. The cancellation of the spontaneous magnetization is explicitly demonstrated in a particular class of models. The same calculation produces a general formula for the electric charge density induced by the magnetic flux – an effect previously discussed in the literature for axionic domain walls. The distribution of the induced charge is calculated in specific models.

# 1 Introduction

The possibility that domain walls that could have existed in the early universe could also be related to generation of a primordial magnetic field correlated at large distances [1] has been recently discussed in the literature [2-5]. The discussed models are based on the idea that fermions, coupled to the field forming the wall, develop a spontaneous magnetization perpendicular to the wall. Although it is a simple exercise in general physics to show that a uniform magnetization of an infinite domain wall does not produce magnetic field [6], the phenomenon of magnetization of the wall is interesting on its own.

The claims to a non-zero magnetic moment of the ground state of a fermion field coupled to the wall in (3+1) dimensions are inferred from the behavior in a (2+1) dimensional QED of Dirac fermions with a definite sign of the mass term  $m$ . Namely, in certain calculations [2, 7, 8, 9] of the energy of the ground state of the fermion field in an external magnetic field  $B$ , it is claimed that the total energy contains a linear in magnetic field term proportional to  $mB$ , which corresponds to a spontaneous magnetization proportional to the mass parameter  $m$  (including the sign). For a Dirac fermion field coupled to a domain wall in (3+1) dimensions the quantization of the motion perpendicular to the wall splits the fermion system into an infinite set of modes, each corresponding to a (2+1) dimensional QED with its own parameter  $m$ . If the phase of the fermion mass varies across the wall, the set of positive values of  $m$  differs from that of the negative values of  $m$ . Therefore in this picture it might at least be not obvious that the overall magnetization cancels after summation over modes corresponding to positive and negative values of  $m$ .

It is nevertheless quite easy to argue that the cancellation necessarily takes place and the magnetization of the fermion field ground state at the wall in (3+1) dimensions is strictly zero. Indeed, the Lagrangian density for the fermions with the phase of the mass  $\mu$  depending on the coordinate  $z$  perpendicular to the wall can be generally written as

$$L = \bar{\psi} \left[ i (\partial_\alpha - i A_\alpha) \gamma^\alpha - \mu_1(z) - i \mu_2(z) \gamma^5 \right] \psi \quad (1)$$

with  $\mu_1$  and  $\mu_2$  being respectively the real and imaginary parts of  $\mu(z)$ , and  $A$  standing for the electromagnetic field potential and absorbing the charge  $e$  in the normalization of the field. The variation of  $\mu_2$  breaks the  $P$  and  $CP$  parities. However the  $C$  parity is manifestly conserved with both  $\mu_1$  and  $\mu_2$  being  $C$ -even. Therefore after the fermions are integrated out the energy of the system as a function of  $A$  in the  $C$ -even background of  $\mu(z)$  cannot contain odd powers of the  $C$ -odd field  $A$ . In particular the energy cannot be linear in the magnetic field  $B$ . The same argument holds also for the situation where the fermions are assumed to

possess anomalous magnetic and/or electric dipole moment since both such interactions also conserve the  $C$  parity.

This simple general  $C$  parity argument is clearly sufficient for excluding the possibility of spontaneous magnetization of the fermion vacuum in a domain wall background. However due to the existence in the literature of claims to the contrary [3, 9] it is quite instructive to demonstrate explicitly the vanishing of the linear in the magnetic field  $B_z = B$  term in the energy (as well as of all the odd terms) at least in a specific model. Moreover the  $C$  parity argument certainly allows dependence of the energy on *even* powers of the electromagnetic field, which gives rise to the most interesting phenomenon of appearance of electric charge density once a magnetic flux is applied across the wall. This phenomenon is related to the well known coupling of a pseudoscalar field to the electromagnetic invariant  $\mathbf{E} \cdot \mathbf{B}$ . For slowly varying with  $z$  mass term, one can approximate  $\mu(z)$  around a given point  $z_0$  as  $\mu(z) = \mu(z_0) + \delta\mu(z)$ , and treating the varying part  $\delta\mu(z)$  as small perturbation, find the term proportional to  $\mathbf{E} \cdot \mathbf{B}$  in the energy density  $w$  from the well known triangle graph as

$$\delta w = \frac{1}{4\pi^2} \frac{\mu_2 \delta\mu_1 - \mu_1 \delta\mu_2}{\mu_1^2 + \mu_2^2} \mathbf{E} \cdot \mathbf{B} . \quad (2)$$

The charge density is then found from variation with respect to the potential  $A_0$ . For the magnetic field in the  $z$  direction,  $B_z = B$ , one finds

$$\langle \bar{\psi}(z) \gamma^0 \psi(z) \rangle = -\frac{\delta w}{\delta A_0} = \frac{B}{4\pi^2} \frac{d}{dz} \arctan \left( \frac{\mu_2(z)}{\mu_1(z)} \right) . \quad (3)$$

The total charge is then given by the total flux of the magnetic field through the wall,  $F = \int dx dy B_z$  and the difference of the phases of the mass term  $\mu_1 + i\mu_2$  at two infinities in  $z$ ,  $\Delta\Phi = \arctan(\mu_2/\mu_1)|_{z \rightarrow +\infty} - \arctan(\mu_2/\mu_1)|_{z \rightarrow -\infty}$ ,

$$Q = \int dx dy dz \langle \bar{\psi}(z) \gamma^0 \psi(z) \rangle = \frac{F \Delta\Phi}{4\pi^2} . \quad (4)$$

The induced charge is a direct analog of the (generally fractional) fermionic charge of a kink in (1+1) dimensional models [10, 11]. This phenomenon in (3+1) dimensions was considered [12] in connection with a magnetic monopole (dyon) traversing an axionic domain wall, in which process the net change (reversal) of the magnetic flux across the wall results in charge exchange between the dyon and the wall<sup>1</sup>.

---

<sup>1</sup>The paper [12] contains a reference to an unpublished communication with H. Georgi and J. Polchinski, who apparently had also interpreted the effect of the triangle graph in terms of an induced charge in a magnetic field. However, as they both kindly corresponded to me, they had never pursued this issue beyond an unwritten remark. The monopole – axion wall charge exchange was also later discussed by I. Kogan [13].

In the specific model, considered in the present paper, the calculation of the induced charge density automatically comes along with the calculation of the (eventually vanishing) spontaneous magnetization. It will be shown that, as expected on general grounds, the relation (4) between the total induced charge and the total magnetic flux does not depend on the specific shape of the dependence of the phase  $\Phi$  on  $z$ . However the *distribution* of the induced charge density in  $z$  does depend on the specific rate of variation of the mass parameter  $\mu(z)$ , and generally differs from that given by the equation (3) that is justified in the limit where the rate of variation can be considered as slow. One can notice in this connection that the charge distribution may be of a greater physical relevance than the total charge, since barring the existence of monopoles it is physically impossible to produce a net magnetic flux through an infinite or closed wall. Therefore the total charge has to be zero, while the distribution of the density of charge can be nontrivial.

The further material in this paper is organized as follows. In Sec.2 the considered class of models is described as well as some properties of the relevant operators corresponding to the motion in the  $x - y$  plane in magnetic field and to motion in the  $z$  direction in the domain wall background are discussed. In Sec.3 the dependence on  $B$  of the energy of the ground state of the fermion field is calculated and the vanishing of all odd terms in the expansion in  $B$  is demonstrated including the vanishing of the spontaneous magnetization. In Sec.4 a general expression for the induced charge density is presented, and Sec.5 contains the calculation of this density in the limit of slowly varying phase as well as a discussion of the topological nature of the total induced charge. An explicit calculation of the distribution of the induced charge in two particular sample situations is considered in Sec.6. Finally, a general discussion and a summary of results are presented in Sec. 7.

## 2 The model and the relevant operators

The simplifying assumption in the class of models to be considered here is that the real part  $\mu_1$  of the mass term is fixed and nonzero, while the imaginary part  $\mu_2$  depends on  $z$ . To the best of the author's knowledge, a model of such type with  $\mu_2$  being proportional to a spinless field  $\phi$ ,  $\mu_2 = g\phi$ , with the field varying as  $\phi(z) = v \tanh(m_\phi z/2)$  across the domain wall, was first suggested by T.D. Lee and G.C. Wick [14] as a model of spontaneous breaking of the  $CP$  symmetry. The fermion spectrum and scattering states in the presence of the domain wall in this model were studied in Ref. [15].

In a generic case of a non-zero change of  $\mu_2$  between the infinities in  $z$  and a non-zero

constant  $\mu_1$ , one can assume for definiteness without further loss of generality that  $\mu_1$  is positive, and  $\mu_2(z)$  changes from a negative value at  $z \rightarrow -\infty$  to a positive one at  $z \rightarrow +\infty$ . Also for definiteness it is assumed here that a uniform positive magnetic field  $B$  is applied in the  $z$  direction. In what follows the gauge for the electromagnetic field is fixed in a standard way such that the vector potential for the field  $B$  is given by  $A_x = 0$ ,  $A_y = Bx$ . Adopting also the standard representation for the  $\gamma$  matrices, the one-particle Hamiltonian, corresponding to the Lagrangian (1) takes the following form

$$H = \begin{pmatrix} \mu_1 & 0 & i P^\dagger & -i R \\ 0 & \mu_1 & i R^\dagger & i P \\ -i P & -i R & -\mu_1 & 0 \\ i R^\dagger & -i P^\dagger & 0 & -\mu_1 \end{pmatrix}, \quad (5)$$

and  $H^2$  has the diagonal form:

$$H^2 = \text{diag} \left( \mu_1^2 + P^\dagger P + R R^\dagger, \mu_1^2 + P P^\dagger + R^\dagger R, \mu_1^2 + P P^\dagger + R R^\dagger, \mu_1^2 + P^\dagger P + R^\dagger R \right). \quad (6)$$

These formulas make use of the following notation for the operators describing respectively the Landau quantization of the motion in the  $x - y$  plane and the quantization of motion along the  $z$  axis:

$$R = \partial_x + Bx + p_y, \quad R^\dagger = -\partial_x + Bx + p_y, \quad (7)$$

and

$$P = \partial_z + \mu_2(z), \quad P^\dagger = -\partial_z + \mu_2(z). \quad (8)$$

The quantity  $p_y$  is the value of the conserved momentum in the  $y$  direction (as a consequence of the chosen gauge condition). The energy levels, determined by the eigenvalues of  $R^\dagger R$  and  $R R^\dagger$  do not depend on  $p_y$  and the degeneracy number is well known to be given by  $BS/(2\pi)$ , where  $S$  is the normalization area in the  $x - y$  plane. In what follows this degeneracy factor will be explicitly accounted for, and the value of  $p_y$  set to  $p_y = 0$  in the definition of the operators (7).

According to Eq.(6) the spectrum of one-particle energies is determined by eigenvalues of the operators  $R^\dagger R$ ,  $R R^\dagger$ ,  $P^\dagger P$ , and  $P P^\dagger$ . These spectra exhibit two separate structures found in supersymmetric quantum mechanics, well known for the operators  $R$  and  $R^\dagger$  and also recently used for the  $P$  type operators in connection with kinks in (1+1) dimensional models [16]. Namely, if the issue of boundary conditions in  $z$  is ignored, one would naively conclude that the spectra of eigenvalues of  $P^\dagger P$  and  $P P^\dagger$  coincide except for an extra zero

eigenvalue (under the adopted sign conventions) of  $P^\dagger P$ . Indeed, let  $v_k$ ,  $k = 1, 2, \dots$ , be the normalized eigenfunction corresponding to the (necessarily positive) eigenvalue  $\lambda_k^2$  of the positive operator  $PP^\dagger$ , so that

$$PP^\dagger v_k = \lambda_k^2 v_k . \quad (9)$$

Applying the operator  $P^\dagger$  to both sides of this equation, one finds that the function

$$u_k = P^\dagger v_k / \lambda_k \quad (10)$$

is the normalized eigenfunction of the operator  $P^\dagger P$  with the same eigenvalue  $\lambda_k^2$ . Applying the operator  $P$  to both sides of the latter relation and using (9), one find the inverse of the relation (10)

$$v_k = P u_k / \lambda_k . \quad (11)$$

This construction does not work however for the zero mode  $u_0$  of  $P^\dagger P$ , satisfying the equation  $P u_0 = 0$  (BPS state). The explicit form of the normalizable function  $u_0(z)$  is readily found from the definition (8):

$$u_0(z) = \text{const} \cdot \exp \left( - \int^z \mu_2(\tilde{z}) d\tilde{z} \right) . \quad (12)$$

The operators  $R^\dagger$  and  $R$  coincide, up to normalization, with the creation and annihilation operators for a harmonic oscillator, and their spectra are the text-book ones: the spectrum of eigenvalues of  $RR^\dagger$  is given by  $2 B n$  with  $n = 1, 2, \dots$ , while that of  $R^\dagger R$  is given by the same simple expression, however also including  $n = 0$ .

This discussion of the properties of the operators involved in the Hamiltonian in Eq.(5) is helpful in considering the spectrum of the one-particle energies. In particular, one can separately consider each eigenmode of the motion in the  $z$  direction as a (2+1) dimensional fermion system. Then the eigenmodes of the operator  $P^\dagger P$  correspond to such system with positive mass parameter  $m$ :  $m = \sqrt{P^\dagger P + \mu_1^2}$ , while the eigenmodes of  $PP^\dagger$  correspond to negative  $m$ :  $m = -\sqrt{PP^\dagger + \mu_1^2}$ . For the former modes the negative energy spectrum of Landau levels is given by  $-\sqrt{P^\dagger P + \mu_1^2 + 2 B n}$  and includes  $n = 0$ , while for the latter ones the negative energies are given by  $-\sqrt{PP^\dagger + \mu_1^2 + 2 B n}$  excluding  $n = 0$ . Since all the non-zero eigenvalues of  $P^\dagger P$  and  $PP^\dagger$  coincide, one might very naively conjecture that their effects in magnetization cancel and the net result is given by only one ‘unpaired’ zero mode of the operator  $P^\dagger P$ . Such conjecture however would be false, since the summation over the modes is generally divergent, and one should perform a proper calculation with a proper regularization. Also the discretization of the continuum spectra of the operators  $P^\dagger P$  and

$PP^\dagger$  requires imposing conditions at the boundaries of large but finite bounding box in  $z$ . These conditions generally are not satisfied by the relations (10) and (11), and there arises a splitting of the spectra of continuum modes of the operators  $P^\dagger P$  and  $PP^\dagger$  which should be accounted for in a proper calculation.

### 3 Dependence of the fermion vacuum energy on $B$ .

For a full regularized calculation of the energy of the ground state of the fermion field we use here the standard four dimensional technique with the Pauli-Villars regularization procedure. The latter regularization preserves the gauge invariance and, importantly for the discussed problem, the Lorentz covariance<sup>2</sup>. This amounts to introducing the regulator fermion field  $\Psi$  with large but finite mass, so that the mass term for  $\Psi$  can be written as  $M + i\mu_2 \gamma^5$  instead of  $\mu_1 + i\mu_2 \gamma^5$  for the ‘physical’ fermion  $\psi$ , and treat the loop with the regulator field with an extra minus sign<sup>3</sup>. The regularized expression for the total energy  $W$  then reads as

$$W = i \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \left\{ \text{Tr} \ln [iD_\alpha \gamma^\alpha - \mu_1 - i\mu_2(z) \gamma^5] - \text{Tr} \ln [iD_\alpha \gamma^\alpha - M - i\mu_2(z) \gamma^5] \right\} = i \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \text{Tr} \int_{\mu_1}^M dm [iD_\alpha \gamma^\alpha - m - i\mu_2(z) \gamma^5]^{-1}, \quad (13)$$

where  $iD_\alpha \gamma^\alpha = \gamma^0 p_0 - \boldsymbol{\gamma} \cdot (\mathbf{p} + \mathbf{A})$  with the spatial momentum  $\mathbf{p}$  understood as operator, and the trace running over the spinor indices and the spatial variables.

The inverse of the Dirac operator in the last expression in Eq.(13) can be readily found by the usual multiplication of the numerator and the denominator by  $iD_\alpha \gamma^\alpha + m - i\mu_2(z) \gamma^5$ . After taking into account the previously mentioned degeneracy in the momentum  $p_y$ , the result can be written as

$$W = i \frac{B S}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \text{Tr} \int_{\mu_1}^M dm \begin{pmatrix} p_0 + m & 0 & -i P^\dagger & i R \\ 0 & p_0 + m & -i R^\dagger & -i P \\ -i P & -i R & -p_0 + m & 0 \\ i R^\dagger & -i P^\dagger & 0 & -p_0 + m \end{pmatrix} [p_0^2 - H^2(m)]^{-1}, \quad (14)$$

---

<sup>2</sup>The vanishing of spontaneous magnetization can in fact be viewed as due to the possibility of using a regularization preserving the Lorentz symmetry.

<sup>3</sup>Only one regulator field is indicated here for simplicity of expressions, whereas the full regularization of the vacuum energy requires additional regulator fields. However for the considered here effects in the energy one regulator is in fact sufficient.

where  $H^2(m)$  is the same diagonal matrix as in Eq.(6) with  $\mu_1$  being replaced by  $m$ . The trace over the spinor variables leaves only the contribution of the diagonal factors in Eq.(14), which thus can be rewritten as

$$W = i \frac{B S}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \int_{\mu_1}^M dm \left\{ \text{Tr} \left[ \frac{p_0 \gamma^0}{p_0^2 - H^2(m)} \right] + \text{Tr} \left[ \frac{m}{p_0^2 - H^2(m)} \right] \right\} . \quad (15)$$

The first term in the curly braces is manifestly odd in  $p_0$ , thus the integration over  $p_0$  from  $-\infty$  to  $\infty$  yields zero. (The integral can be verified to be finite, thus no further regularization, that potentially could invalidate the symmetry argument, is needed.) The second term after explicitly performing the trace over the spinor variables takes the form

$$W = -i \frac{B S}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \int_{\mu_1}^M dm m \text{Tr} \left[ \left( \frac{1}{-p_0^2 + m^2 + P^\dagger P + R R^\dagger} + \frac{1}{-p_0^2 + m^2 + P^\dagger P + R^\dagger R} \right) + \left( \frac{1}{-p_0^2 + m^2 + P P^\dagger + R R^\dagger} + \frac{1}{-p_0^2 + m^2 + P P^\dagger + R^\dagger R} \right) \right] . \quad (16)$$

Here in braces are grouped together the terms with the same ordering of  $P$  and  $P^\dagger$  and different order of  $R$  and  $R^\dagger$ . Each of the two expressions in the braces has the general form

$$\text{Tr} \left( \frac{1}{X + R R^\dagger} + \frac{1}{X + R^\dagger R} \right) , \quad (17)$$

where the operator  $X$  does not depend on the magnetic field. It is now easy to show that the expression (17) contains only odd powers in its expansion in  $B$  and thus that the expansion in  $B$  of the vacuum energy described by Eq.(16) contains only even powers, as required by the  $C$  invariance. Indeed, applying the formula  $x^{-1} = \int_0^\infty \exp(-\beta x) d\beta$  and using the described spectra of the operators  $R R^\dagger$  and  $R^\dagger R$ , one can perform the trace over the space of the latter operators and find the expression (17) in the form

$$\int_0^\infty d\beta \coth(\beta B) \text{Tr} \exp(-\beta X) , \quad (18)$$

which is manifestly odd in  $B$ <sup>4</sup>.

This concludes the proof by an explicit calculation of the absence of odd powers of the magnetic field  $B$  in the expansion of the energy of the fermion vacuum.

---

<sup>4</sup>Clearly the first term of the expansion of the expression in Eq.(18), proportional to  $B^{-1}$ , results in a divergent expression, which corresponds to the divergence of the vacuum energy at  $B = 0$ . This is the only place where additional Pauli-Villars regulators are formally required. The subsequent terms of the expansion however give a finite difference  $W(B) - W(0)$  that is even in  $B$ .



## 4 Fermion charge density induced by magnetic field

We now proceed to a calculation of the electric charge induced by the external magnetic field  $B$ . The general formula for the charge density  $\rho(z) = \langle \bar{\psi}(z) \gamma^0 \psi(z) \rangle$  is obtained from the variational derivative of the energy in Eq.(13) with respect to  $A_0(z)$  at  $A_0 = 0$  and reads as

$$\rho(z) = -\frac{i}{S} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \left\langle z \left| \text{Tr} \left\{ \gamma_0 \left[ iD_\alpha \gamma^\alpha - \mu_1 - i\mu_2(z) \gamma^5 \right]^{-1} \right\} \right| z \right\rangle . \quad (19)$$

The charge density is finite thus eliminating the need for considering the contribution of the Pauli-Villars regulator. The trace here runs over the spinor indices and over the space of the operators  $R$  and  $R^\dagger$ . The  $z$  dependence is thus left ‘untraced’, as is indicated in Eq.(19) by the diagonal matrix element of the remaining (after ‘tracing out’ the spinor variables and the motion in the  $x - y$  plane)  $z$  dependent operator in the coordinate representation. Performing also the trace over the  $z$  dependence, i.e. calculating the integral over  $z$ , would yield the total induced charge  $Q$ . Using the inverse of the Dirac operator as in Eq.(14) one can calculate the trace over the spinor indices and rewrite the latter formula for  $\rho$  as

$$\begin{aligned} \rho(z) = & \mu_1 \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \left\langle z \left| \text{Tr} \left[ \left( \frac{1}{p_0^2 + \mu_1^2 + P^\dagger P + R^\dagger R} - \frac{1}{p_0^2 + \mu_1^2 + P^\dagger P + R R^\dagger} \right) \right. \right. \right. \\ & \left. \left. \left. - \left( \frac{1}{p_0^2 + \mu_1^2 + P P^\dagger + R^\dagger R} - \frac{1}{p_0^2 + \mu_1^2 + P P^\dagger + R R^\dagger} \right) \right] \right| z \right\rangle . \end{aligned} \quad (20)$$

In the latter expression also the Wick rotation  $p_0 \rightarrow ip_0$  is done and the part of the integrand that is odd in  $p_0$  is discarded. The trace over the space of the operators  $R$  and  $R^\dagger$  in Eq.(20) is finite and trivial:

$$\text{Tr}_{(R)} \left( \frac{1}{X + R^\dagger R} - \frac{1}{X + R R^\dagger} \right) = \frac{1}{X} . \quad (21)$$

Using this formula in Eq.(20) one finds

$$\rho(z) = \mu_1 \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \left\langle z \left| \frac{1}{p_0^2 + \mu_1^2 + P^\dagger P} - \frac{1}{p_0^2 + \mu_1^2 + P P^\dagger} \right| z \right\rangle . \quad (22)$$

One can notice the absence of higher than linear terms in the expansion of the charge density in powers of  $B$ , which is a consequence of the relation (21). It can be also observed that up to the factor  $B/(2\pi)$  the expression (22) coincides with similarly calculated fermionic charge in a (1+1) dimensional theory with  $z$  dependent mass term  $\mu_1 + i\mu_2(z) \gamma^5$ . This relates the present calculations to the problem of fermion charge of kinks [10, 11].

## 5 Charge density for slowly varying phase and the topological nature of the total charge

We now consider few specific cases of the dependence of  $\mu_2$  on  $z$ , in which the charge density can be calculated from Eq.(22) either in a closed form, or at least ‘in quadratures’. Also the topological nature of the total charge is to be addressed in this section.

The first case to be considered is the one where the rate of variation of the parameter  $\mu_2$  can be considered as slow. In this case in order to calculate the charge density at a point  $z_0$ , one can approximate  $\mu_2(z)$  near  $z_0$  by the linear dependence:  $\mu_2(z) \approx \mu_2(z_0) + (z - z_0) \mu'_2(z_0)$  and also expand the difference of the Green’s functions in Eq.(22) to the linear order in  $\mu'_2(z_0)$ :

$$\begin{aligned} \rho(z_0) &= 2 \mu_1 \mu'_2(z_0) \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \left\langle z_0 \left| \left( -\partial_z^2 + p_0^2 + \mu_1^2 + \mu_2^2(z_0) \right)^{-2} \right| z_0 \right\rangle \\ &= 2 \mu_1 \mu'_2(z_0) \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0 dp_z}{(2\pi)^2} \left( p_z^2 + p_0^2 + \mu_1^2 + \mu_2^2(z_0) \right)^{-2} \\ &= \frac{B}{4\pi^2} \frac{\mu_1 \mu'_2(z_0)}{\mu_1^2 + \mu_2^2(z_0)} = \frac{B}{4\pi^2} \frac{d}{dz} \arctan \left( \frac{\mu_2(z)}{\mu_1} \right) \Big|_{z=z_0}, \end{aligned} \quad (23)$$

where the final expression matches that in Eq.(3) in the considered case of a constant  $\mu_1$ . Clearly, for this calculation to be justified the characteristic length of variation of  $\mu_2(z)$  should be much longer than the ‘local’ Compton wavelength  $(\mu_1^2 + \mu_2^2(z_0))^{-1/2}$ .

Let us address now the statement that the total integral over the density,  $Q = \int_{-\infty}^{\infty} \rho(z) dz$ , is in fact topological, i.e. it is determined only by the limiting values of  $\mu_2(z)$  at the infinities and does not depend on the specific shape of the function  $\mu(z)$  at finite  $z$ . From Eq.(22) the total integral can be written as a trace:

$$\begin{aligned} Q &= \mu_1 \frac{B S}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \text{Tr} \left( \frac{1}{p_0^2 + \mu_1^2 + P^\dagger P} - \frac{1}{p_0^2 + \mu_1^2 + P P^\dagger} \right) \\ &= \mu_1 \frac{B S}{4\pi} \text{Tr} \left( \frac{1}{\sqrt{\mu_1^2 + P^\dagger P}} - \frac{1}{\sqrt{\mu_1^2 + P P^\dagger}} \right). \end{aligned} \quad (24)$$

In the latter expression the traces of the individual operator terms in the braces are infinite. For this reason one cannot calculate them separately, each in its own basis. If each trace were finite, one could calculate the trace of the first operator in the basis of the eigenfunctions  $u_k$  of the operator  $P^\dagger P$  and the trace of the second operator in the basis of the eigenfunctions  $v_k$  of the operator  $P P^\dagger$  (cf. equations (9 - 11)). In that case the

difference would arise only from the ‘extra’ zero mode (12) of the operator  $P^\dagger P$ . In view of the divergence of each of the traces only the trace of the whole operator in braces in Eq.(24) can be calculated in an arbitrary complete basis. Also the continuum spectrum of the operators should be discretized by choosing a large bounding box in  $z$ : conventionally from  $-L/2$  to  $+L/2$ , where  $L$  is sufficiently large for the asymptotic values of  $\mu_2(z)$  to set on, and by imposing boundary conditions at  $z = \pm L/2$ . Consider, for instance, the case where the trace in Eq.(24) is calculated in the basis of the eigenfunctions  $v_k$  of the operator  $PP^\dagger$  with an (anti)periodic condition at the boundaries of the box. Then the eigenfunctions  $u_k$  of the operator  $P^\dagger P$  found from Eq.(10) do not satisfy the same condition since  $\mu_2$  takes different values at the infinities. This effect thus gives rise to a splitting of the eigenvalues in the two terms in the braces in Eq.(24). The latter effect however vanishes, if  $\mu_2(z)$  takes the same value at both infinities. Also in this case the operator  $P^\dagger P$  no longer has the zero mode described by Eq.(12), and the spectra of the two discussed terms in Eq.(24) coincide in the same basis of (anti)periodic eigenfunctions. Thus the total charge vanishes, if there is no net variation of  $\mu_2$ :  $\mu_2(-\infty) = \mu_2(\infty)$ .

The latter observation in combination with Eq.(23) is in fact sufficient to conclude that the total charge depends only on the difference of the limiting values of  $\mu_2$ . Indeed, consider a large box, defined by the length  $L$ , and let the difference  $\Delta = \mu_2(L/2) - \mu(-L/2)$  be nonzero. Consider now an *extended* system, where the box is continued from  $z = L/2$  to  $z = \tilde{L}$ , in such a way that the length of the extension  $\tilde{L} - L/2$  is arbitrarily large, and the behavior of  $\mu_2(z)$  in the extension is chosen ‘by hand’ such that it is a slow function smoothly interpolating  $\mu_2$  from its fixed value at  $z = L/2$  back to  $\mu_2(-L/2)$  at  $z = \tilde{L}$ . In other words, the net variation over the extension  $\tilde{\Delta} = \mu_2(\tilde{L}) - \mu_2(L/2)$  is  $\tilde{\Delta} = -\Delta$ . The net change of  $\mu_2$  over all the extended box is then zero. Thus the total charge of the extended system is also zero. The total charge however is a sum of the one inside the original box,  $Q$ , and that in the extension,  $\tilde{Q}$ . Thus  $Q = -\tilde{Q}$ . On the other hand, the charge  $\tilde{Q}$  can be calculated in the limit of slowly varying  $\mu_2(z)$  from the formula (23). Therefore one arrives at the equation (4) for the charge  $Q$ , independently of the details of the behavior of  $\mu_2(z)$  at intermediate  $z$  within the original bounding box.

## 6 Charge density in sample models

Here we consider explicit calculations of  $\rho(z)$  from the formula (22) in sample models of the dependence of  $\mu_2(z)$  on  $z$  in order to illustrate that the distribution of charge generally does

not follow the slow variation limit described by Eq.(23) while the total charge is of course given by the universal formula (4).

A limit that is maximally opposite to that of a slow variation is where  $\mu_2$  changes as a step function at a point in  $z$ . Here we consider the situation where  $\mu_2$  changes at  $z = 0$  from a negative constant value  $\mu_2(z) = -\mu_2$  at  $z < 0$  to a positive one,  $\mu_2(z) = \mu_2$  at  $z > 0$ . The Green's functions in Eq.(22) can be found explicitly by the standard method of matching at  $z = 0$ :

$$\begin{aligned} \left\langle z \left| \frac{1}{p_0^2 + \mu_1^2 + P^\dagger P} \right| z \right\rangle &= \frac{1}{2q} \left[ 1 + \frac{\mu_2}{q - \mu_2} \exp(-2q|z|) \right], \\ \left\langle z \left| \frac{1}{p_0^2 + \mu_1^2 + PP^\dagger} \right| z \right\rangle &= \frac{1}{2q} \left[ 1 - \frac{\mu_2}{q + \mu_2} \exp(-2q|z|) \right], \end{aligned} \quad (25)$$

where  $q = \sqrt{p_0^2 + \mu_1^2 + \mu_2^2}$ . The charge density is then given 'in quadratures' by the formula

$$\rho(z) = \mu_1 \mu_2 \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{\exp\left(-2\sqrt{p_0^2 + \mu_1^2 + \mu_2^2}|z|\right)}{(p_0^2 + \mu_1^2)}, \quad (26)$$

which, as one could expect, describes an exponential distribution around the discontinuity point  $z = 0$  over the range  $(\mu_1^2 + \mu_2^2)^{-1/2}$  that is the Compton wavelength of the fermion. For the total charge the integration over  $z$  and then over  $p_0$  is elementary, and the result matches the general formula (4).

It can be also noted, in connection with this example, that the non-exponential term in individual Green's functions in Eq.(25) would lead to a divergence upon integration over  $p_0$ . However this term cancels in the difference, leaving the expression for  $\rho(z)$  finite. This illustrates the behavior discussed in general terms in the previous section.

Another class of specific situations where the Green's functions are readily calculable in a closed form and the charge density can be found explicitly is presented by the Lee-Wick model [14]. In this model the term  $\mu_2$  in the fermion mass arises from the interaction with a neutral field  $\phi$ :  $\mu_2 = g\phi$ , with  $g$  being the Yukawa coupling. The  $\phi^4$  self-interaction of the field  $\phi$  is assumed to lead to the domain wall solution

$$\phi(z) = v \tanh(\lambda z/2), \quad (27)$$

which interpolates between the two vacuum states (domains) with  $\phi = \pm v$ , and where  $\lambda$  is the constant of the self-interaction, such that the mass of the  $\phi$  bosons in either of the vacua is  $m_\phi = \lambda v$ .

The behavior of the fermion states in such background depends on the ratio  $g/\lambda$  [15]. Clearly, the case  $g \gg \lambda$  corresponds to the limit of slow variation of  $\mu_2$ , while the opposite case,  $g \ll \lambda$ , corresponds to the approximation of an abrupt change in  $\mu_2$ , provided that in the latter case  $\mu_1$  is also assumed to be small in comparison with  $m_\phi$ . For a generic value of the ratio  $g/\lambda$  the operators  $P^\dagger P$  and  $PP^\dagger$  correspond to the solvable potentials of the form  $\text{const}/\cosh^2(\lambda z/2)$ , and the Green's functions can be found in terms of hypergeometric functions. In a situation where  $2g/\lambda$  is an integer:  $2g/\lambda = N$ , both potentials are non reflecting, and the algebra is greatly simplified, since the relevant hypergeometric functions collapse to (Jacobi) polynomials in  $\tanh(\lambda z/2)$  of the power  $N$  for  $P^\dagger P$  and  $N - 1$  for  $PP^\dagger$ . It can be also noted, that the case of  $N = 2$  in the considered class of models would correspond to a supersymmetric model, albeit with  $\mu_1 = 0$ , which would take us beyond the assumptions adopted in the present paper. Here for illustrative purposes we consider only the most algebraically simple case of  $N = 1$ .

For  $\lambda = 2g$ , the operator  $PP^\dagger$  corresponds to a constant potential:  $PP^\dagger = -\partial_z^2 + g^2 v^2$ , and the corresponding Green's function is especially simple:

$$G_{(PP^\dagger)}(z_1, z_2; p_0^2 + \mu_1^2) \equiv \left\langle z_1 \left| \frac{1}{p_0^2 + \mu_1^2 + PP^\dagger} \right| z_2 \right\rangle = \frac{\exp(-q|z_1 - z_2|)}{2q}, \quad (28)$$

with  $q = \sqrt{p_0^2 + \mu_1^2 + g^2 v^2}$ . The Green's function for the operator  $P^\dagger P$  can then be found [16], using the relations (10) and (12) for its eigenfunctions<sup>5</sup>:

$$\begin{aligned} G_{(P^\dagger P)}(z_1, z_2; p_0^2 + \mu_1^2) &\equiv \left\langle z_1 \left| \frac{1}{p_0^2 + \mu_1^2 + P^\dagger P} \right| z_2 \right\rangle = \sum_{k=0}^{\infty} \frac{u_k(z_1) u_k^*(z_2)}{p_0^2 + \mu_1^2 + \lambda_k^2} \\ &= \sum_{k=1}^{\infty} P_{z_1}^\dagger \frac{v_k(z_1) v_k^*(z_2)}{\lambda_k^2 (p_0^2 + \mu_1^2 + \lambda_k^2)} P_{z_2} + \frac{u_0(z_1) u_0(z_2)}{p_0^2 + \mu_1^2} \\ &= \frac{1}{p_0^2 + \mu_1^2} \left[ P^\dagger G_{(PP^\dagger)}(z_1, z_2; 0) P - P^\dagger G_{(PP^\dagger)}(z_1, z_2; p_0^2 + \mu_1^2) P \right] + \frac{u_0(z_1) u_0(z_2)}{p_0^2 + \mu_1^2}. \end{aligned} \quad (29)$$

Applying in this formula the operators  $P^\dagger$  and  $P$  as indicated to the expression (28), and using the zero mode from Eq.(12):  $u_0(z) = \sqrt{gv/2}/\cosh(gvz)$ , one finds that at coinciding points the Green's function (29) is given by

$$G_{(P^\dagger P)}(z, z; p_0^2 + \mu_1^2) = \frac{1}{2q} + \frac{g^2 v^2}{2q(p_0^2 + \mu_1^2) \cosh^2(gvz)}, \quad (30)$$

where, again,  $q = \sqrt{p_0^2 + \mu_1^2 + g^2 v^2}$ .

---

<sup>5</sup>The individual Green's functions are finite, thus each can be expanded in the basis of the corresponding eigenfunctions.

Upon substitution of the results from the equations (28) and (30) to Eq.(22) for the charge density, and after the integration over  $p_0$ , the final expression for the distribution of charge in this model is found in a remarkably simple explicit form:

$$\rho(z) = \frac{g v}{2 \cosh^2(g v z)} \frac{Q}{S} , \quad (31)$$

where  $Q = B S \arctan[gv/(\mu_1^2 + g^2 v^2)]/(2\pi^2)$  is the total induced charge, in agreement with the general formula (4).

## 7 Discussion

The explicit calculation described in the sections 2 and 3 illustrates the implementation of the general statement about the absence of spontaneous magnetization based on the  $C$  parity. In the concrete calculation the expected cancellation occurs due the fact that it is possible to regularize the theory while manifestly preserving the charge conjugation symmetry. Alternatively, as already noted, this cancellation can be viewed as the manifest Lorentz invariance of the regularized theory, since a nonzero spontaneous magnetization would certainly imply a breaking of the Lorentz symmetry [8]. In other words, neither the  $C$  symmetry nor the Lorentz one are anomalous. The same symmetries preclude an appearance of a spontaneous magnetization also in models where the fermions are assumed to have anomalous magnetic or/and electric dipole moments, including the cases [4] where the fermions are neutron-like, i.e. with zero charge and nonzero dipole moments.

The really existing effect, allowed by both these symmetries, is the fermion charge density induced by an external magnetic field applied across the domain wall, which can be traced to the well-known anomaly in the axial current. The total induced charge is of a topological nature, and is determined, according to Eq.(4), by the total magnetic flux through the wall and the asymptotic values of the fields in the model far from the wall. The distribution of the induced charge however, given by the general formula (22) depends on the details of the profile of the wall, as discussed in the sections 5 and 6. This effect, in principle, can be relevant in detailed analyses of phenomena at the walls in the early universe – either for flat domain walls, which could exist during some epoch, or for dynamics near the walls of bubbles during a first order phase transition.

The arguments, based on the  $C$  parity and/or the Lorentz symmetry, do not directly apply to an asymmetric state with a net overall fermion charge, e.g. to the early universe

with the baryon asymmetry, that is neither  $C$  nor Lorentz symmetric. In this case a spontaneous magnetization of the wall is generally allowed, and is proportional to the asymmetry parameter of the considered state [6]. It should however be clearly understood that even a magnetized domain wall does not produce a magnetic field, which is a simple consequence of the classical Maxwell's equations [6]. Therefore domain walls could not be a source of a correlated at cosmological distances magnetic field in the early universe, although their electromagnetic properties, like the charge distribution induced by a magnetic field, could generally be of importance in other aspects of dynamics of the early universe.

## 8 Acknowledgements

I am thankful to A. Larkin, M. Shifman and A. Vainshtein for illuminating discussions and to H. Georgi and J. Polchinski for correspondence related to the history of the problem of the induced charge. This work is supported in part by DOE under the grant number DE-FG02-94ER40823.

## References

- [1] D. Grasso and H.R. Rubinstein, Padua Univ. Report DFPD-00-TH-35, Sep 2000; [astro-ph/0009061]
- [2] A. Iwazaki, Phys. Lett. **B 406** (1997) 304; Phys. Rev. **D 56** (1997) 2435.
- [3] P. Cea and L. Tedesco, Phys. Lett. **B 450** (1999) 61.
- [4] M.M. Forbes and A.R. Zhitnitsky, Phys. Rev. Lett. **85** (2000) 5268; [hep-ph/0004051]
- [5] M.M. Forbes and A.R. Zhitnitsky, Univ. of British Columbia Report, Feb 2001; [hep-ph/0102158]
- [6] M.B. Voloshin, Phys. Lett. **B491** (2000) 311; [hep-ph/0007123]
- [7] P. Cea, Phys. Rev. **D 32** (85) 2785; Phys. Rev. **D 34** (86) 3229.
- [8] Y. Hosotani, Phys. Lett. **B319** (1993) 332; Phys. Rev. **D 51** (1995) 2022.
- [9] P. Cea and L. Tedesco, Bari Univ. Report BARI-TH-395-00, Oct 2000; [hep-ph/0010095]

- [10] R. Jackiw and C. Rebbi, Phys. Rev. **D 13** (1976) 3398.
- [11] J. Goldstone and F. Wiczek, Phys. Rev. Lett. **47** (1981) (986).
- [12] P. Sikivie, Phys. Lett. **B137** (1984) 353.
- [13] I.I. Kogan, Phys. Lett. **B299** (1993) 16.
- [14] T.D. Lee and G.C. Wick, Phys. Rev. **D9** (1974) 2291.
- [15] M.B. Voloshin, Yad.Fiz. **21** (1975) 1331.
- [16] M. Shifman, A. Vainshtein, and M. Voloshin, Phys. Rev. **D59** (1999) 045016; [hep-th/9810068]